Vector Spaces, Span, and Basis

1. **Are the vectors**

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

   linearly independent?

2. **Do the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis in $\mathbb{R}^3$?**

3. **Decompose the vector**

$$\mathbf{u} = \begin{pmatrix} 7 \\ -5 \\ -7 \end{pmatrix}$$

   onto the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

4. **Are the coefficients you found in Problem 3 the only ones that decompose $\mathbf{u}$ over the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$?**

5. **Verify that the vectors**

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

   are mutually orthogonal.

6. **Are the above vectors orthonormal?**

7. **Make the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ into an orthonormal set.**

8. **Are the vectors $\hat{\mathbf{v}}_1$, $\hat{\mathbf{v}}_2$, and $\hat{\mathbf{v}}_3$ a basis for $\mathbb{R}^3$? How about the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$?**

9. **Decompose the vector**

$$\mathbf{u} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$$

onto the basis vectors $\hat{\mathbf{v}}_1$, $\hat{\mathbf{v}}_2$, and $\hat{\mathbf{v}}_3$.

10. **Given vectors**

$$\mathbf{x}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

calculate the vector that is the projection of $\mathbf{x}_2$ onto $\mathbf{x}_1$.

11. **Using your answer to Question 6, find a vector nearest to $\mathbf{x}_2$ that is orthogonal to $\mathbf{x}_1$.**

12. CHALLENGE: **Look back at our proof of the theorem for decomposing a vector over an orthonormal basis using dot products. Can we use dot products to decompose over a basis where all vectors are orthogonal but not normalized?**

Solutions

1. Are the vectors

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

linearly independent?

Let's form a matrix $\mathbf{V}$ using the vectors as columns.

$$\mathbf{V} = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & -2 \\ 2 & 0 & -1 \end{pmatrix}$$

This matrix has three rows and three columns. If all three columns are linearly independent, the matrix will have rank three.

$$\begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & -2 \\ 2 & 0 & -1 \end{pmatrix} \xrightarrow{-R_1} \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & -2 \\ 2 & 0 & -1 \end{pmatrix}$$

$$\xrightarrow{-2R_1 \to R_3} \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & -2 \\ 0 & 4 & 1 \end{pmatrix}$$

$$\xrightarrow{-4R_2 \to R_3} \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 9 \end{pmatrix}$$

$$\xrightarrow{(1/9)R_3} \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

The reduced form of $\mathbf{V}$ shows that $\text{rank}(\mathbf{V}) = 3$, so the vectors are linearly independent.

2. Do the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis in $\mathbb{R}^3$?

Yes. The space $\mathbb{R}^3$ has dimension three, and we have three vectors. We know from the previous question that these three vectors are linearly independent. Combined, these facts prove that the vectors are a basis for $\mathbb{R}^3$. Since they are a basis, we also know that the vectors span $\mathbb{R}^3$.

3. Decompose the vector

$$\mathbf{u} = \begin{pmatrix} 7 \\ -5 \\ -7 \end{pmatrix}$$

onto the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

We are looking for scalars a_1, a_2, and a_3 such that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{u}$$

which is equivalent to solving the linear system

$$\mathbf{Va} = \mathbf{u}$$

where $\mathbf{V}$ is a matrix with columns $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$; and $\mathbf{a}$ is a vector of the scalars a_1, a_2, and a_3:

$$\begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & -2 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 7 \\ -5 \\ -7 \end{pmatrix}$$

We can solve this by Gaussian elimination to find that $a_1 = -2$, $a_2 = 1$, and $a_3 = 3$.

4. Are the coefficients you found in Problem 3 the only ones that decompose u over the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$?

Yes. Since the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are a basis, the decomposition of any vector onto them is unique.

5. Verify that the vectors

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

are mutually orthogonal.

The vectors are mutually orthogonal if every vector is orthogonal to every other vector. We can check orthogonality by computing dot products.

$$\begin{aligned} \mathbf{v}_1 \cdot \mathbf{v}_2 &= (-1)(1) + (1)(-1) + (1)(2) = 0 \\ \mathbf{v}_1 \cdot \mathbf{v}_3 &= (-1)(1) + (1)(1) + (1)(0) = 0 \\ \mathbf{v}_2 \cdot \mathbf{v}_3 &= (1)(1) + (-1)(1) + (2)(0) = 0 \end{aligned}$$

6. Are the above vectors orthonormal?

We know the vectors are mutually orthogonal. For the vectors to be an orthonormal set, every vector needs to be a unit vector.

$$\begin{aligned} \|\mathbf{v}_1\| &= \sqrt{(-1)^2 + 1^2 + 1^2} = \sqrt{3} \neq 1 \\ \|\mathbf{v}_2\| &= \sqrt{1^2 + (-1)^2 + (2)^2} = \sqrt{6} \neq 1 \\ \|\mathbf{v}_3\| &= \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2} \neq 1 \end{aligned}$$

None of the vectors are unit vectors. We knew the vectors were not an orthonormal set after computing the first magnitude.

7. Make the vectors v_1, v_2, and v_3 into an orthonormal set.

We showed that the three vectors are mutually orthogonal. All that remains is to normalize each vector by dividing by its magnitude.

$$\hat{\mathbf{v}}_1 = (1/\sqrt{3})\mathbf{v}_1$$
$$\hat{\mathbf{v}}_2 = (1/\sqrt{6})\mathbf{v}_2$$
$$\hat{\mathbf{v}}_3 = (1/\sqrt{2})\mathbf{v}_3$$

8. Are the vectors $\hat{\mathbf{v}}_1$, $\hat{\mathbf{v}}_2$, and $\hat{\mathbf{v}}_3$ a basis for $\mathbb{R}^3$? How about the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$?

A basis in $\mathbb{R}^3$ requires three vectors. The vectors $\hat{\mathbf{v}}_1$, $\hat{\mathbf{v}}_2$, and $\hat{\mathbf{v}}_3$ are mutually orthogonal, so they are also linearly independent (see the beginning of § 11.5). Any three linearly independent vectors form a basis in $\mathbb{R}^3$. This argument requires only that the vectors be mutually orthogonal, not orthonormal, so the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ also form a basis in $\mathbb{R}^3$.

9. Decompose the vector

$$\mathbf{u} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$$

onto the basis vectors $\hat{\mathbf{v}}_1$, $\hat{\mathbf{v}}_2$, and $\hat{\mathbf{v}}_3$.

We are looking for coefficients a_1, a_2, and a_3 such that

$$\mathbf{u} = a_1\hat{\mathbf{v}}_1 + a_2\hat{\mathbf{v}}_2 + a_3\hat{\mathbf{v}}_3$$

Our basis is orthonormal so we can apply the shortcut formula using dot products.

$$\begin{aligned}
a_1 &= \mathbf{u} \cdot \hat{\mathbf{v}}_1 \\
&= \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} \cdot \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \\
&= (-3 + 4 - 2)/\sqrt{3} = -1/\sqrt{3} \\
a_2 &= \mathbf{u} \cdot \hat{\mathbf{v}}_2 \\
&= \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} \cdot \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \\
&= (3 - 4 - 4)/\sqrt{6} = -5/\sqrt{6} \\
a_3 &= \mathbf{u} \cdot \hat{\mathbf{v}}_3 \\
&= \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\
&= (3 + 4 + 0)/\sqrt{2} = 7/\sqrt{2}
\end{aligned}$$

We can confirm that

$$\begin{aligned}
&a_1\hat{\mathbf{v}}_1 + a_2\hat{\mathbf{v}}_2 + a_3\hat{\mathbf{v}}_3 \\
&= -\frac{1}{\sqrt{3}} \times \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} - \frac{5}{\sqrt{6}} \times \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \frac{7}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\
&= -\frac{1}{3} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} - \frac{5}{6} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \frac{7}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} = \mathbf{u}
\end{aligned}$$

10. **Given vectors**

$$\mathbf{x}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

calculate the vector that is the projection of $\mathbf{x}_2$ onto $\mathbf{x}_1$.

$$\begin{aligned} \mathrm{proj}_{\mathbf{x}_1}(\mathbf{x}_2) &= \frac{\mathbf{x}_2 \cdot \mathbf{x}_1}{\mathbf{x}_1 \cdot \mathbf{x}_1}\mathbf{x}_1 \\ &= \frac{1 \times -2 + 1 \times 1}{-2 \times -2 + 1 \times 1}\mathbf{x}_1 \\ &= \begin{pmatrix} 2/5 \\ -1/5 \end{pmatrix} \end{aligned}$$

11. **Using your answer to Question 6, find a vector nearest to $\mathbf{x}_2$ that is orthogonal to $\mathbf{x}_1$.**

The vector $\mathbf{x}_2 - \mathrm{proj}_{\mathbf{x}_1}(\mathbf{x}_2)$ is orthogonal to the vector $\mathbf{x}_1$.

$$\begin{aligned} \mathbf{x}_2 - \mathrm{proj}_{\mathbf{x}_1}(\mathbf{x}_2) &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2/5 \\ -1/5 \end{pmatrix} \\ &= \begin{pmatrix} 3/5 \\ 6/5 \end{pmatrix} \end{aligned}$$

To verify, we compute the dot product between our new vector and $\mathbf{x}_1$.

$$\begin{pmatrix} 3/5 \\ 6/5 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} = -6/5 + 6/5 = 0$$

12. Challenge: **Look back at our proof of the theorem for decomposing a vector over an orthonormal basis using dot products. Can we use dot**

products to decompose over a basis where all vectors are orthogonal but not normalized?

We use the normality of the vectors after we show that

$$\mathbf{u} \cdot \hat{\mathbf{v}}_i = a_i \hat{\mathbf{v}}_i \cdot \hat{\mathbf{v}}_i = a_i \left\|\hat{\mathbf{v}}_i\right\|^2$$

If the basis vectors were orthogonal but not normalized, we would instead have

$$\mathbf{u} \cdot \mathbf{v}_i = a_i \mathbf{v}_i \cdot \mathbf{v}_i = a_i \left\|\mathbf{v}_i\right\|^2$$

Here we are unable to assume that $\left\|\mathbf{v}_i\right\|^2 = 1$ since $\mathbf{v}_i$ is not a normal vector. But we can still solve for the coefficient a_i.

$$a_i = (\mathbf{u} \cdot \mathbf{v}_i)/\left\|\mathbf{v}_i\right\|^2$$

The above formula uses only a dot product and a norm to compute each coefficient. These are far easier calculations than solving the linear system $\mathbf{Va} = \mathbf{u}$ for non-orthogonal basis vectors. The power of orthonormal basis vectors comes from their orthogonality, not their normality.