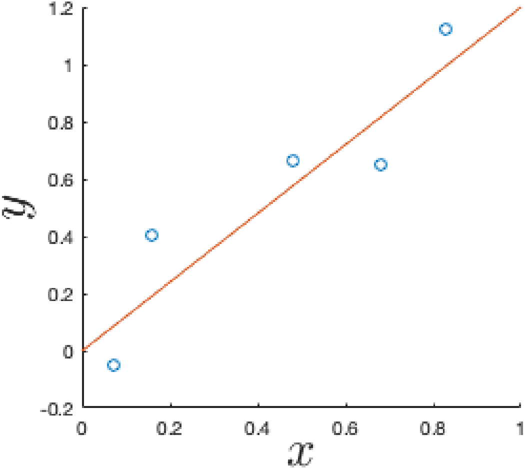
Linear Models II

BIOE 210

Review: A *noisy* linear system

x^{true}	y^{true}
0.07	-0.05
0.16	0.40
0.48	0.66
0.68	0.65
0.83	1.12



A matrix formalism for linear models

Let's write out one equation for each observation of the model $y = \beta_0 + \beta_1 x$.

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$$\begin{pmatrix} -0.05 \\ 0.40 \\ 0.66 \\ 0.65 \\ 1.12 \end{pmatrix} = \begin{pmatrix} 1 & 0.07 \\ 1 & 0.16 \\ 1 & 0.48 \\ 1 & 0.68 \\ 1 & 0.83 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \end{pmatrix}$$

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$$\mathbf{y} = \mathbf{X}\beta + \epsilon$$

Solving the linear system

A few points about $\mathbf{y} = \mathbf{X}\beta + \epsilon$:

- The unknowns are β, not $\mathbf{X}$.
- The coefficient matrix $\mathbf{X}$ is called the *model matrix*.
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The solution to this system that minimizes the errors in ϵ is

$$\beta = \mathbf{X}^{+}\mathbf{y}$$

where $\mathbf{X}^{+}$ is the *pseudoinverse* of $\mathbf{X}$.

The intercept

The linear model

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Or, in vector form

$$\mathbf{y} = \begin{pmatrix} \mathbf{1} & \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} + \epsilon.$$

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Why would we not want an intercept?

- When all inputs $\mathbf{x}_i = 0$, the response $y = \beta_0$.
- If we know our system has zero response without an input, we don't include an intercept.
- This is rare, so most models include an intercept.

Prediction vs. Inference

Prediction uses a model to find the response of inputs we've never seen before.

Inference uses a model to understand what inputs determine the response.

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We feed the training data back into the model to find the residuals:

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We quantify the accuracy using the *root mean squared error* of the residuals.

$$\text{RMSE} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(y_i^{\text{true}} - y_i^{\text{pred}}\right)^2}$$

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Example: A model to predict pulse rate has RMSE of 12 bpm. If the model predicts 68 bpm for a patient, the 95% confidence interval is

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Remember: If you transformed your model, the RMSE will be in the transformed units!

Model inference

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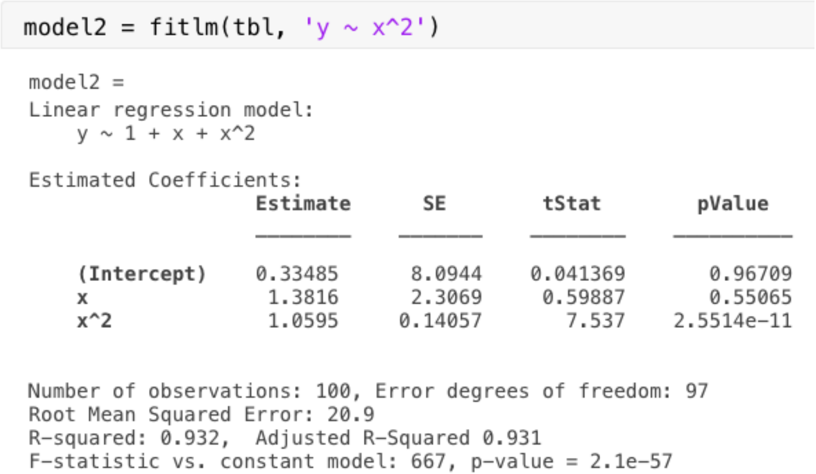
- A unit change in variable $\mathbf{x}_1$ would *decrease* the response by 3.6 units.
- A unit change in $\mathbf{x}_2$ would *increase* the response by 0.8 units.

How sure are we of the effect sizes?



```
model2 = fitlm(tbl, 'y ~ x^2')
```

```
model2 =
Linear regression model:
    y ~ 1 + x + x^2

Estimated Coefficients:
                   Estimate      SE        tStat       pValue
                   ________    _______    ________    __________

    (Intercept)    0.33485      8.0944    0.041369       0.96709
    x               1.3816      2.3069     0.59887       0.55065
    x^2             1.0595     0.14057       7.537    2.5514e-11


Number of observations: 100, Error degrees of freedom: 97
Root Mean Squared Error: 20.9
R-squared: 0.932,  Adjusted R-Squared 0.931
F-statistic vs. constant model: 667, p-value = 2.1e-57
```

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- A low *p*-value indicates that an effect of this size was unlikely to occur randomly.
- It also means the confidence interval *excludes* zero, so we reject the hypothesis that the true effect is zero.

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- A low *p*-value indicates that an effect of this size was unlikely to occur randomly.
- It also means the confidence interval *excludes* zero, so we reject the hypothesis that the true effect is zero.
- It does **not** mean that the effect is practically significant or important. (That's up to the effect size.)

RESEARCH ARTICLE

Marital satisfaction and break-ups differ across on-line and off-line meeting venues

John T. Cacioppo, Stephanie Cacioppo, Gian C. Gonzaga,
Elizabeth L. Ogburn, and Tyler J. VanderWeele

PNAS June 18, 2013 110 (25) 10135-10140; https://doi.org/10.1073/pnas.1222447110

For respondents categorized as currently married at the time of the survey, we examined marital satisfaction. Analyses indicated that currently married respondents who met their spouse on-line reported higher marital satisfaction (M = 5.64, SE = 0.02, n = 5,349) than currently married respondents who met their spouse off-line (M = 5.48, SE = 0.01, n = 12,253; mean difference = 0.18, $F_{(1,17,601)} = 46.67$, $P < 0.001$*).*

Interactions

Imagine we're modeling the response (y) from two input variables, $\mathbf{x}_1$ and $\mathbf{x}_2$. The simplest model is

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What is there is another effect that depends on both $\mathbf{x}_1$ and $\mathbf{x}_2$? This is an **interaction** between $\mathbf{x}_1$ and $\mathbf{x}_2$.

How do we model interactions?

We model the interaction of $\mathbf{x}_1$ and $\mathbf{x}_2$ using the product of these variables.

$$y = \beta_1 \mathbf{x}_1 + \beta_2 \mathbf{x}_2 + \beta_{12} \mathbf{x}_1 {:} \mathbf{x}_2 + \epsilon$$

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The coefficient β_{12} is the effect size of the interaction.

Why do we multiply $\mathbf{x}_1$ and $\mathbf{x}_2$? There are at least two ways to interpret this term.

The coded factor interpretation

Often we set up design matrices using **coded variables**. If we're testing the variable at two levels, we code the variable as "on/off" ($\{0, 1\}$) or "low/high" ($\{-1, +1\}$).

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on/off → interaction when both "on"

$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_1 : \mathbf{x}_2$
0	0	0
0	1	0
1	0	0
1	1	1

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high/low → interaction when both "high" or both "low"

$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_1{:}\mathbf{x}_2$
-1	-1	$+1$
-1	$+1$	-1
$+1$	-1	-1
$+1$	$+1$	$+1$

The augmented slope interpretation

We can also interpret the interaction as one variable changing the effect of the other variable.

$$\begin{aligned} y &= \beta_1 \mathbf{x}_1 + \beta_2(\mathbf{x}_1){:}\mathbf{x}_2 + \epsilon \\ &= \beta_1 \mathbf{x}_1 + (\beta_2 + \beta_{12}\mathbf{x}_1){:}\mathbf{x}_2 + \epsilon \\ &= \beta_1 \mathbf{x}_1 + \beta_2 \mathbf{x}_2 + \beta_{12}\mathbf{x}_1{:}\mathbf{x}_2 + \epsilon \end{aligned}$$

Things to remember about interactions

- Interaction are modeled as the product of variables.
- The interaction effect is “above and beyond" the independent effects (synergy/super-additivity, antagonism/sub-additivity).
- Higher-order interactions are possible (e.g. $\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3$), but these are rare.